Equations of motion

- Models of ocean circulation are all based on the “equations of motion”.
  - Only in simple cases the equations of motion can be solved analytically, usually they must be solved numerically.

Equations of motion

- Newton's second law of motion:

$$\text{force} = \text{mass} \times \text{acceleration}$$

or

$$\text{acceleration} = \frac{\text{force}}{\text{mass}}$$

applied to a fluid moving over the surface of the Earth

Equations of motion

- In dealing with moving fluids, consider forces acting "per unit volume":

$$\text{acceleration} = \frac{\text{force per unit volume}}{\text{density}}$$

$$= \frac{1}{\rho} \times \text{force per unit volume}$$

- Apply in three dimensions x, y, z at right angles to one another

- Current velocities in x-, y- and z-directions:

$$u, v, w$$

- Accelerations (rates of change of velocity with time) in x-, y- and z-directions:

$$\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$$

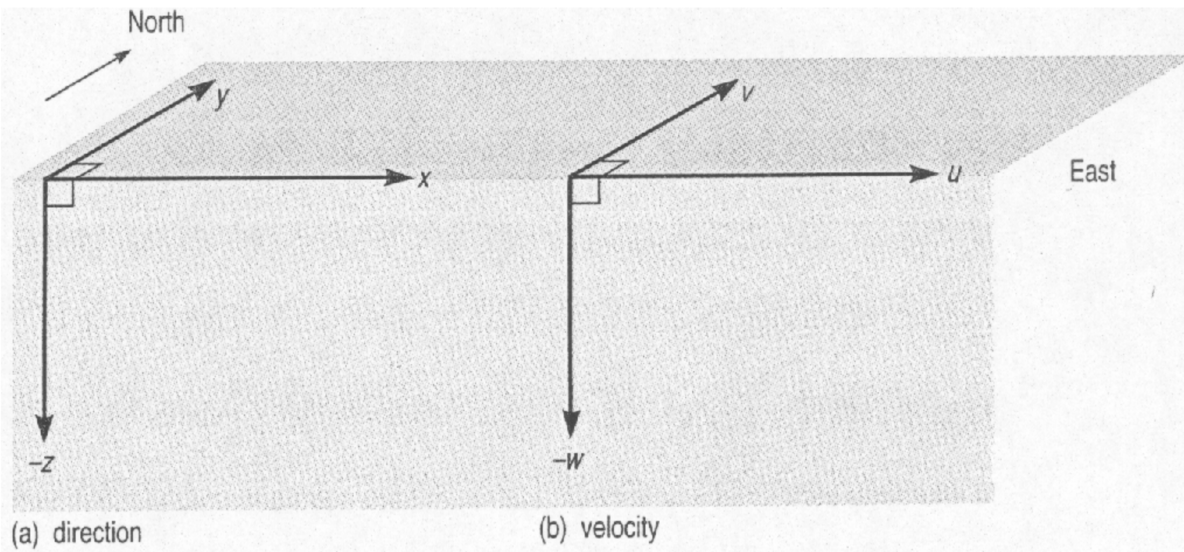

Figure 4.15 from Open University (1989)

- What forces might lead to acceleration in the horizontal x- and/or y-directions, and therefore need to be included in the equations of motion for those directions?

Based on Section 4.2.3 of Open University (1989)

- What forces might lead to acceleration in the horizontal x- and/or y-directions, and therefore need to be included in the equations of motion for those directions?
  - Coriolis force
  - horizontal pressure gradient force
  - wind stress and other frictional forces

- The Coriolis force is proportional to the sine of the latitude.

- For a particle of mass m moving with speed u, the Coriolis force is given by:

$$\text{Coriolis force} = m \times 2\Omega \sin\phi \times u$$

where Ω is the angular velocity of the Earth about its axis and ϕ is latitude.

- Using the abbreviation

$$\text{Coriolis parameter} = f = 2\Omega \sin\phi$$

the expression for the Coriolis force becomes:

$$\text{Coriolis force} = m f u$$

- As a force per unit volume:

$$\text{Coriolis force per unit volume} = \rho f u$$

- Equation of motion for the x-direction:

$$\frac{du}{dt} = \frac{1}{\rho} \times \left(\begin{array}{c} \text{horizontal} \\ \text{pressure} \\ \text{gradient force in} \\ \text{the } x\text{-direction} \end{array} + \begin{array}{c} \text{Coriolis force} \\ \text{resulting in} \\ \text{motion in the} \\ x\text{-direction} \end{array} + \begin{array}{c} \text{other forces} \\ \text{related to} \\ \text{motion in the} \\ x\text{-direction} \end{array} \right)$$

- Similarly for the y-direction
  - left-hand side: dv/dt
  - right-hand side: replace "x-direction" by "y-direction"

- Equations of motion in mathematical terms:

$$\frac{du}{dt} = \frac{1}{\rho} \times \left(- \frac{dp}{dx} + \rho f v + F_x \right)$$

$$\frac{dv}{dt} = \frac{1}{\rho} \times \left(- \frac{dp}{dy} - \rho f u + F_y \right)$$

acceleration

pressure gradient force

Coriolis force

contributions from other forces

- Equations of motion in mathematical terms:

$$\frac{du}{dt} = \frac{1}{\rho} \times \left(- \frac{dp}{dx} + \rho f v + F_x \right)$$

$$\frac{dv}{dt} = \frac{1}{\rho} \times \left(- \frac{dp}{dy} - \rho f u + F_y \right)$$

- F_x and F_y may include
  - wind stress
  - friction
  - tidal forcing

- Why does the equation for flow in the x-direction have $\rho f v$ as the Coriolis term rather than $\rho f u$, and vice versa for flow in the y-direction?

Based on Section 4.2.3 of Open University (1989)

- Why does the equation for flow in the x-direction have $\rho f v$ as the Coriolis term rather than $\rho f u$, and vice versa for flow in the y-direction?
  - Because the Coriolis force acts at right angles to the current (the Coriolis force acting in the x-direction is proportional to the velocity in the y-direction and vice versa)

- What is the main force in the vertical or z-direction?

- What is the main force in the vertical or z-direction?
  - The force due to gravity, written as ρg (weight per unit volume)

- Equation of motion for the *z*-direction:

$$\frac{dw}{dt} = \frac{1}{\rho} \times \left(- \frac{dp}{dz} - \rho\, g + F_z \right)$$

acceleration

pressure gradient force

gravitational force

contributions from other forces

- In the ocean, vertical accelerations are generally very small, and *dw/dt* may often be neglected.

Principle of continuity

- Continuity of mass
  - means mass must be conserved
  - is effectively continuity of volume, because seawater is virtually incompressible
  - used in conjunction with equations of motion
  - provides extra constrains

Principle of continuity

- Continuity of volume during flow

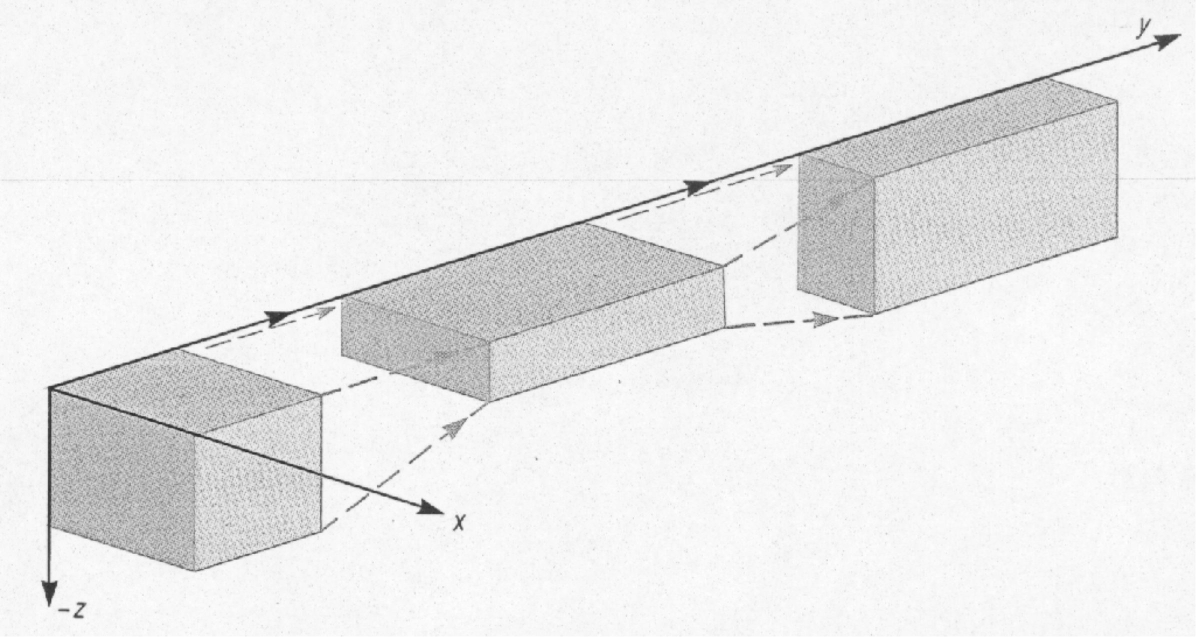

Figure 4.16 from Open University (1989)

- How does the flow pattern of the subtropical gyre exemplify the principle of continuity?

- Mathematical equation used to express the principle of continuity:

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$$

- Any change in the rate of flow in (say) the x-direction must be compensated for by a change in the rate of flow in the y- and/or z-direction(s).

Based on Section 4.2.3 of Open University (1989)